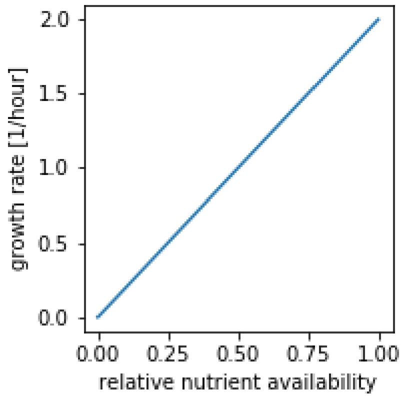
Logistic Growth

In the previous notebook, we explored linear and exponential growth. In both cases, growth goes on forever -- a situation that doesn't typically happen for example since bacteria run out of food. So lets walk through such an example:

- the food initially available is C_0
- division of a bacterium requires x amount of food. Hence there can at most by $N = C_0/x$ new bacteria at the end
- the food remaining after time t is $C(t) = C_0 - x \times (n(t) - n_0)$.
- lets assume the rate of division decreases proportionally with the available food $\frac{C(t)}{C_0\tau}$



With these assumptions and definitions, we find a difference equation

$$\begin{aligned} n(t+\Delta t) &= n(t) + \Delta t \times n(t) \times \frac{C(t)}{C_0\tau} \\ &= n(t) + \Delta t \times \frac{n(t)}{\tau} \times \left(1 - \frac{x(n(t)-n_0)}{C_0}\right) \\ &= n(t) + \Delta t \times \frac{n(t)}{\tau} \times \left(1 - \frac{n(t)-n_0}{N}\right) \end{aligned}$$

Rearranging this into a differential equation in the usual way results in

$$\lim_{\Delta t \to 0} \frac{n(t+\Delta t) - n(t)}{\Delta t} = \frac{dn(t)}{dt} = \frac{n(t)}{\tau} \times \left(1 - \frac{n(t)-n_0}{N}\right)$$

This can be further simplified by realizing that whenever it matters, the $n(t) \gg n_0$ so that we can simply drop n_0 from the right hand side to obtain the standard logistic differential equation:

$$\frac{dn(t)}{dt} = \frac{n(t)}{\tau}\left(1 - \frac{n(t)}{N}\right)$$

Here N is often called carrying capacity.

Before we start solving this equation, lets look at the case $n(t) \ll N$!

In this case, the equation simplifies

$$\frac{dn(t)}{dt} = \frac{n(t)}{\tau}(1 - n(t)/N) \approx \frac{n(t)}{\tau}$$

This is simply exponential growth like we have seen before, but we expect this approximation only to be valid while

$$n(t) \approx n_0 e^{t/\tau} \ll N$$

In [1]:

```
# define function that return derivative
def dndt(n, tau, N):
    return n/tau*(1-n/N)
```

In [2]:

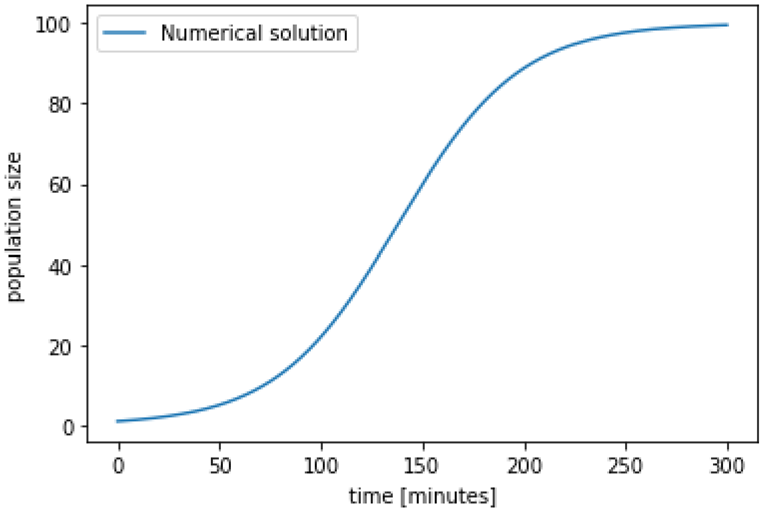
```
tau = 30 # division time of 30 minutes
N = 100
n_0 = 1
n = [n_0]
t = [0]
Delta_t = 0.1
tmax = 10*tau
for i in range(int(tmax//Delta_t)):  # number of steps necessary is tmax divided by step size = tmax/Delta_t
    n.append(n[i] + Delta_t * dndt(n[i],tau,N))
    t.append(t[i] + Delta_t)
```

In [3]:

```
import matplotlib.pyplot as plt
plt.plot(t, n, label=f"Numerical solution")
plt.xlabel("time [minutes]")
plt.ylabel("population size")
plt.legend()
```

Out[3]:

```
<matplotlib.legend.Legend at 0x7f50dc380950>
```



The logistic equation has an exact solution:

$$n(t) = N \frac{e^{t/\tau}}{N/n_0 - 1 + e^{t/\tau}}$$

At $t = 0$ we have $n(0) = n_0$ as it has to be. At very large t, the solution tends to N.

The solution to the logistic equation can be parameterized in different ways and we'll explore these more in the exercises.

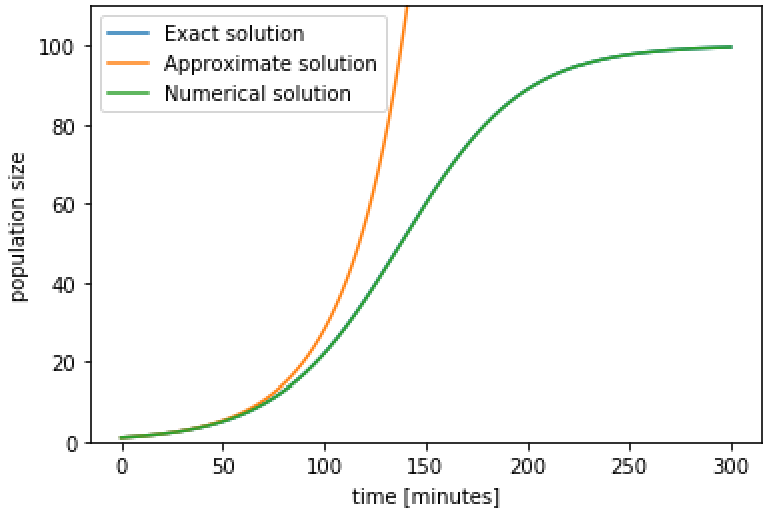
In [4]:

```
import numpy as np
def logistic(t, tau, n_0, N):
    t_arr = np.array(t)
    return N*np.exp(t_arr/tau)/(N/n_0-1+np.exp(t_arr/tau))

plt.plot(t, logistic(t,tau, n_0,N), label="Exact solution")
plt.plot(t, n_0*np.exp(np.array(t)/tau), label="Approximate solution")
plt.plot(t, n, label=f"Numerical solution")
plt.xlabel("time [minutes]")
plt.ylabel("population size")
plt.ylim(0,N*1.1)
plt.legend()
```

Out[4]:

```
<matplotlib.legend.Legend at 0x7f50d191af50>
```



Dig deeper

- change τ, n_0, and N in the above graphs and explore how the results change.
- verify the solution to the logistic equation.
- graph the output on a logarithmic scale.

In [ ]:

In [ ]: