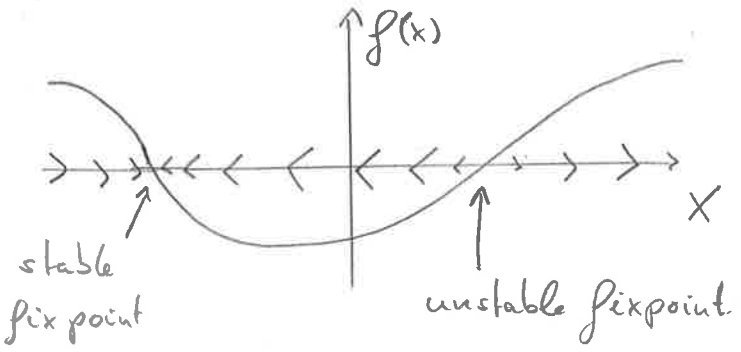
# Dynamical systems (Strogatz)

$$\frac{dx_i}{dt} = f(\{x_i\})$$

$\Rightarrow$ arbitrarily complex dynamics

## One dimensional systems

$$\frac{dx}{dt} = f(x)$$

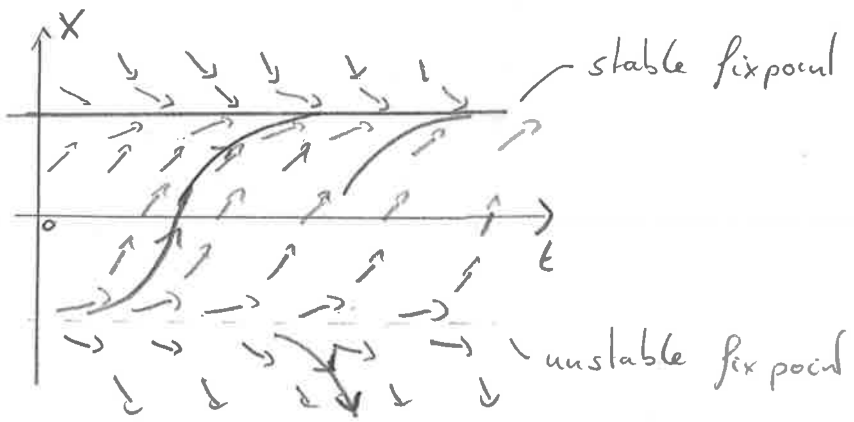



fix point $\Leftrightarrow$ $f(x) = 0$

stable $\Leftrightarrow$ $\frac{df}{dx} < 0$

unstable $\Leftrightarrow$ $\frac{df}{dx} > 0$

qualitative behavior determined by fix points

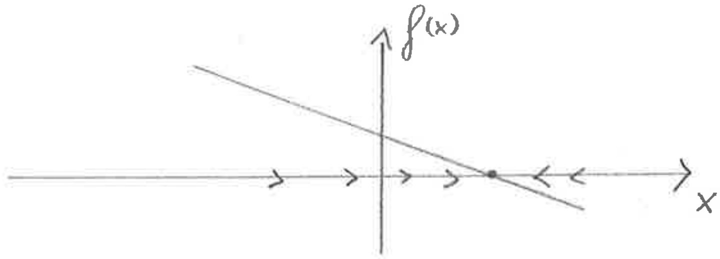



$\rightarrow$ defines a flow in $x$

$\rightarrow$ suggests a way to solve equation numerically

## Example 1

$$\frac{dx}{dt} = \mu - x$$



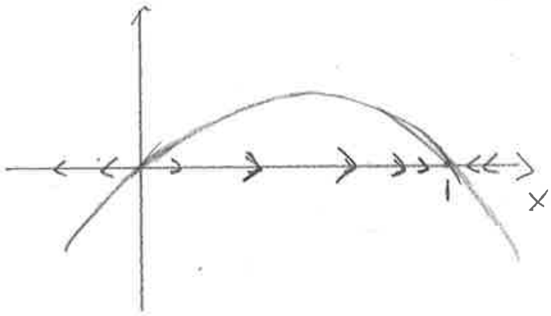

$$x(t) = \mu(1-e^{-t}) + x_0 e^{-t}$$

linear system ⇔ 1 fixpoint

## Example 2

$$\frac{dx}{dt} = a x(1-x)$$

logistic




for $a>0$, $x=1$ stable, $x=0$ unstable

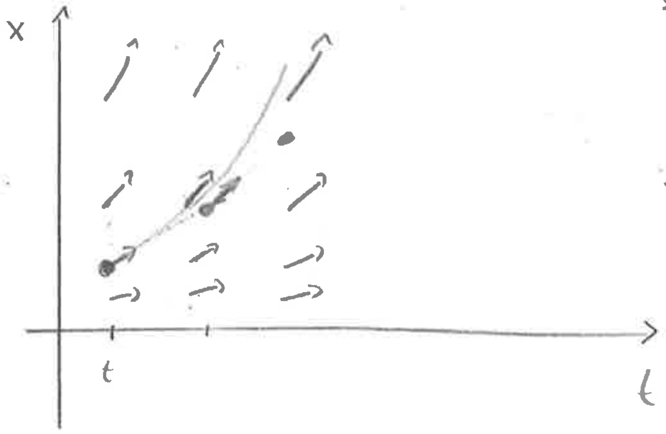
$a<0$ flipped.

$$x(t) = \frac{e^{at}}{C + e^{at}}$$

→ non-linear systems can have multiple fixpoints

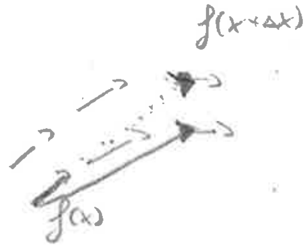
## Excurs

Forward - Euler integration: $x(t+\Delta t) = x(t) + \Delta t\, f(x)$



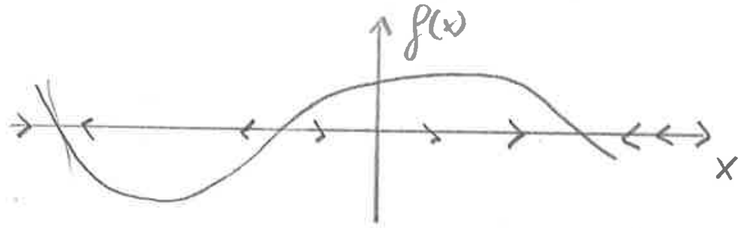

⇒ errors because $f(x)$ changes along the path

→ better use higher order methods




- trial step
- calculate $f(x+\Delta x)$
- "use avg." $\frac{f(x) + f(x+\Delta x)}{2}$
- 4th order Runge-Kutta is standard

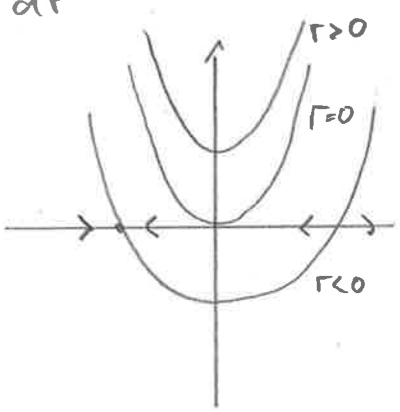
## Bifurcations




$f(x)$ could depend on parameters and time

- fixpoint stay, rates change
- fixpoints move
- fixpoints disappear

$$\frac{dx}{dt} = r + x^2$$




- stable and unstable fixpoint merge
- disappear
- qualitative behavior changes

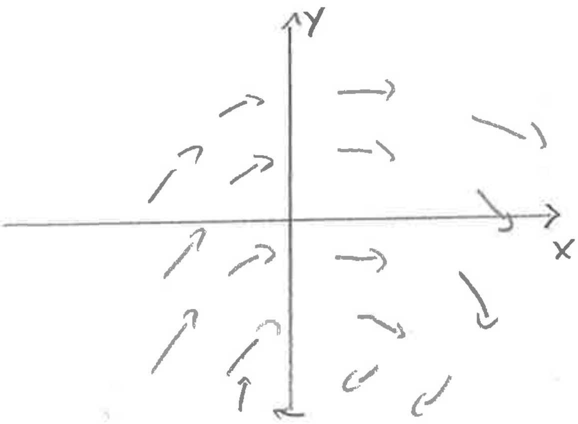
→ Saddle node bifurcation

# Flows in 2 dimension

- in 1d either fixpoint or $\pm\infty$
- much richer in 2d !

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}\right\} \quad \frac{d\vec{x}}{dt} = \vec{F}(\vec{x})$$

a vector at each point in space

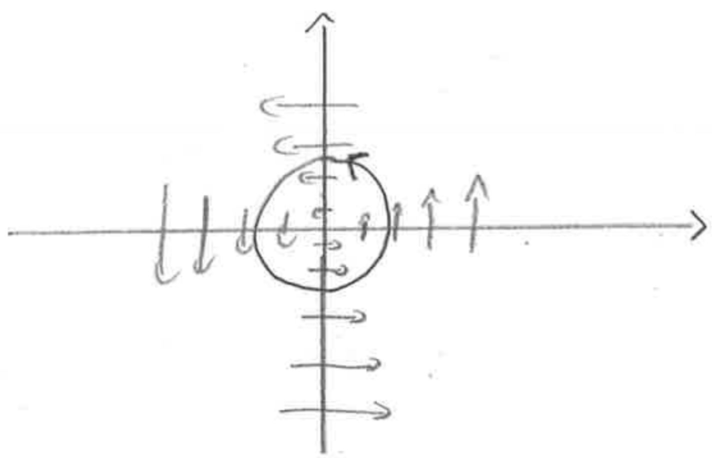



→ circles, spirals, etc

• simplest system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

• oscillator $b=-1$, $c=+1$, $a=d=0$

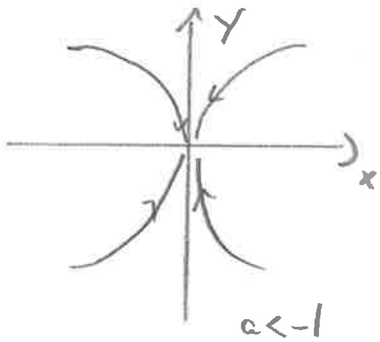


$x = \cos(t)$

$y = \sin(t)$

## uncoupled

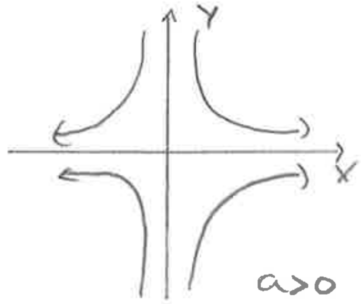
$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \quad \begin{aligned} x &= x_0 e^{at} \\ y &= y_0 e^{-t} \end{aligned}$$




$a < -1$

$\Rightarrow$ approach via the y-axis
x component decays faster

$\Rightarrow$ flip x/y if $-1 > a > 0$




$a > 0$

$\Rightarrow$ unstable in x, stable in y

## general classification

if A is symmetric $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$, we can define

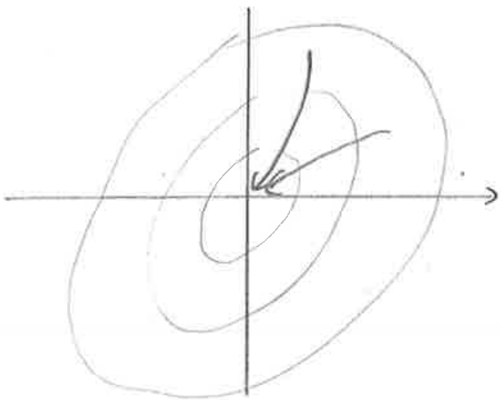
$$V = -\left(\frac{a x^2}{2} + c x y + \frac{a y^2}{2}\right)$$

and find

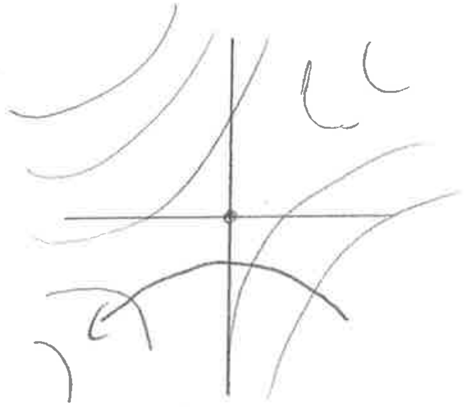
$$\frac{dx}{dt} = -\frac{dV}{dx}$$

$$\frac{dy}{dt} = -\frac{dV}{dy}$$

$\Rightarrow \quad \exists$ potential



dynamics is rolling down a hill



hill, valley, saddle

$\Rightarrow$ depends on the eigenvalues of the matrix

$\Rightarrow$ every symmetric matrix allows rotation such that coordinates are uncoupled

$$\frac{dx'}{dt} = -\lambda_1 x'$$

$$\frac{dy'}{dt} = -\lambda_2 y'$$

<u>non-symmetric $A \Leftrightarrow$ oscillatory component</u>

eigen vectors:

$$\lambda \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix} \Rightarrow \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix} = 0$$

$$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda_{1/2} = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$a+d = \text{Trace}$$

$$ad-bc = \text{determinant}$$