Linear Algebra

$x + y = 7$ # heads $x =$ chicken

$2x + 4y = 22$ # legs $y =$ rabbits

$$\left.\begin{aligned} (1)\quad & ax + by = u \\ (2)\quad & c \cdot x + dy = v \end{aligned}\right\} \text{ general linear } 2\times 2 \text{ equation}$$

$d\cdot(1) - b(2) \Rightarrow (ad - bc)x = ud - vb$

$\Rightarrow x = \dfrac{ud - vb}{ad - bc}$

$c(1) - a(2) \Rightarrow (bc - ad)y = cu - av$

$\Rightarrow y = \dfrac{av - cu}{ad - bc}$

$\Rightarrow$ common denominator $D = ad - bc$

solution exists whenever $D \neq 0$

Matrix - vector notation

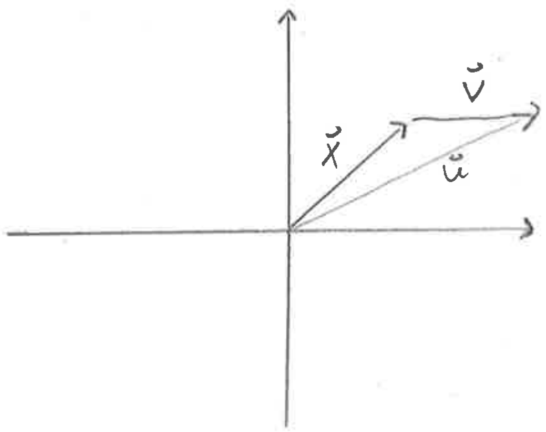
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$M\vec{x} = \vec{u}$$

$$\begin{pmatrix} x \\ u \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\vec{x} = M^{-1}\vec{u}$$

- vector often represent translations



$\vec{x} + \vec{v} = \vec{u}$

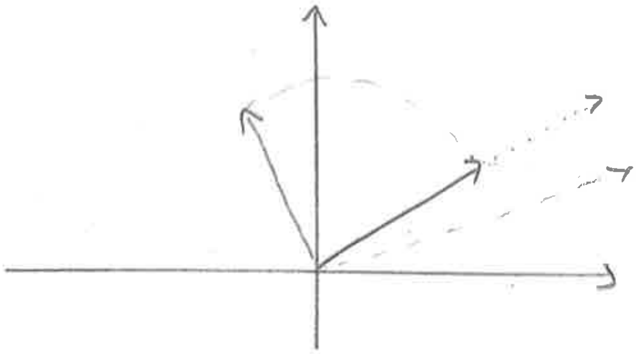
scalar product

$$\vec{x} \cdot \vec{v} = (x_1, x_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = x_1 v_1 + x_2 v_2$$

norm (squared length)

$$\vec{x} \cdot \vec{x} = x_1^2 + x_2^2$$

- matrices are transformations



stretch $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

rotation $\begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin(\Theta) & \cos\Theta \end{pmatrix}$

- matrices map vectors onto other vectors
- can be applied sequentially

$$\vec{v} = M_2 \underbrace{(M_1 \vec{x})}_{\vec{x}\,'} = M_2 M_1 \vec{x}$$

- in general not commutative

$$M_2 M_1 \neq M_1 M_2$$

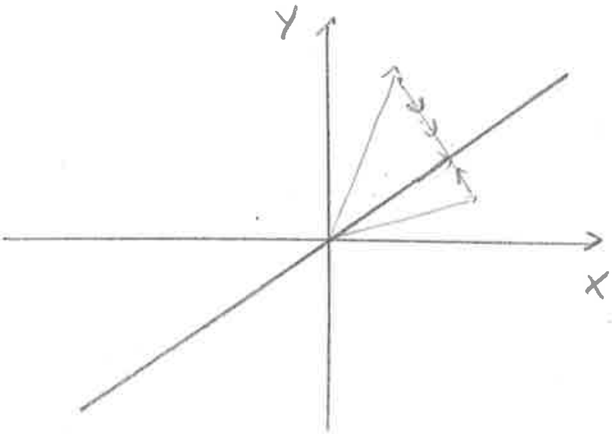
$\Rightarrow$ matrix multiplication

- Matrices need not be square

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = x \qquad (\text{scalar product})$$

- Matrices are invertible if and only if every vector is mapped to a unique other vector



$$\begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix} \Rightarrow \quad y = \lambda x$$

$\Rightarrow$ all points fall on a line

$\Rightarrow$ not invertible

- invertible iff determinant $D \neq 0$
  - matrix has full rank
  - no linearly dependent colums
  - ...

Matrix multiplication

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{12} \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{21} \end{pmatrix}$$

$$e_{11} = \quad \times \quad = m_{11} n_{11} + m_{12} n_{21} \quad \Big| \quad e_{ij} = \sum_k m_{ik} e_{kj}$$

Matrix inverse

$$M\vec{x} = \vec{u} \quad \Rightarrow \quad \vec{x} = M^{-1}\vec{u}$$

$$MM^{-1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \leftarrow \text{identity matrix}$$

Diagonal matrices

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \vec{x} = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix}$$

Eigenvectors and eigenvalues

- special directions with $M\vec{v} = \lambda\vec{v}$ (both λ & $\vec{v}$ unknown)
- in a diagonal matrix, each coordinate direction is an eigen direction

- $$(M - \lambda I)\,\vec{v} = 0 \quad \Leftrightarrow \quad \text{vanishing determinant}$$

- $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow (a-\lambda)(d-\lambda) - bc = \lambda^2 - \underbrace{(a+d)}_{\text{trace}}\lambda + \underbrace{ad-bc}_{D} = 0$$

$$\Rightarrow \lambda_{1/2} = \frac{1}{2}\left[a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)} \right]$$

- knowing $\lambda_{1/2}$ $\Rightarrow$ solve for $\vec{v}_{1/2}$

  $\Rightarrow$ they are only defined up to a multiplicative constant

- real and complex eigenvalues

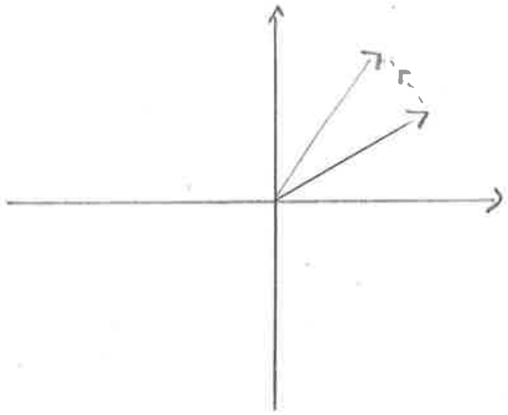
$$\lambda_{1/2} = \frac{1}{2}\left(a+d \pm \sqrt{(a-d)^2 + 4bc} \right)$$

- real if $(a-d)^2 + 4bc > 0$

  $\Rightarrow$ symmetric matrices have real eigenvectors

  $\Rightarrow$ if $bc \geq 0$, eigenvectors are real

- rotations have complex eigenvalues (there can't be an eigendirection with real $\lambda, \vec{v}$



- left/right eigenvectors

$$M\vec{v} = \lambda \vec{v}$$
$${}^{T}M\vec{w} = \lambda \vec{w}$$

- $\vec{v}$ & $\vec{w}$ are identical for symmetric M
- left & right eigenvalues are the same

$$\vec{w}_j M \vec{v}_i = \vec{w}_j \lambda_i \vec{v}_i = \vec{w}_j \lambda_j v_i \quad \Rightarrow \quad \vec{v}_i \vec{w}_j = \delta_{ij} \qquad \lambda_i \neq \lambda_j$$

- Matrices of left & right eigenvector diagonalize a matrix

$$W = \begin{pmatrix} \vec{w}_1 \\ \vdots \\ \vec{w}_n \end{pmatrix} \qquad V = (\vec{v}_1 \dots \vec{v}_n) \quad \Rightarrow \quad WMV = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Solving linear ODE

$$\frac{d\vec{x}}{dt} = M\vec{x}$$

1. express $\vec{x}$ in the right eigen basis

$$\vec{x} = \sum_i a_i(t)\,\vec{v}_i \qquad a_j = \vec{w}_j\,\vec{x} = \sum_i a_i(t)\,\underbrace{\vec{w}_j\,\vec{v}_i}_{\delta_{ij}}$$

2. $$\frac{d\vec{x}}{dt} = M\vec{x} = M\sum_i a_i(t)\,\vec{v}_i = \sum_i \lambda_i\, a_i(t)\, v_i(t)$$

$$\vec{w}_i\,\frac{d\vec{x}}{dt} = \frac{da_i}{dt} = \lambda_i\, a_i(t) \qquad \Rightarrow \qquad a_i(t) = a_i(0)\, e^{\lambda_i t}$$

3. $$\vec{x} = \sum_i a_i(0)\, e^{\lambda_i t}\, \vec{v}_i$$

$\Rightarrow$ sum of exponentials

$Re(\lambda_i) < 0 \Rightarrow$ decay

$Re(\lambda_i) > 0 \Rightarrow$ growth

Complex eigenvalues $\Rightarrow$ rotation